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Fractional powers in Sturmian words

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Résumé

Etant donné un mot infini Sturmian s, nous calculons la fonction $L(m)$ qui donne la longueur du plus long facteur de s ayant pour période m. L'expression de $L(m)$ fait intervenir le développement en fraction continue de l'irrationnel α associé à s. © 2001 Elsevier Science B.V. All rights reserved.

Abstract

Given an infinite Sturmian word s, we calculate the function $L(m)$ which gives the length of the longest factor of s having period m. The expression of $L(m)$ makes use of the continued fraction of the irrational α associated with s. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of the existence of factors of the form $uu \cdots u = u^n$ in long words over a finite alphabet has been initiated for a long time and is the subject of many publications. It is a part of the theory of (un)avoidable patterns in words. The first nontrivial result seems to be due to Thue [13]. Another interesting result of Karhumaki [3] says that the Fibonacci word, which trivially contains 3 th-powers, does not contain 4 th-powers. This result involving a particular Sturmian word was later developed and extended by several authors.

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In particular, in [6] the Sturmian words where the exponent of the powers are bounded are characterized, and in [7] it is proved that the exponents of fractional powers in the Fibonacci word have least upper bound $(5+\sqrt{5})/2$ (a periodic word w with period m is also called a fractional power of its prefix u of length m, with exponent $e=|w|/m$, and is written $w=u^e$).

The fractional powers with exponent $e \geqslant 2$ in the Fibonacci word are described in [9].

Recently, a generalization of these results has been given in the case of Sturmian words which are fixed points of morphisms [14]. The existence of a least upper bound which is also a point of accumulation for the exponents is proved and its value given.

In the present paper (whose results were announced in [10]), given any Sturmian word s, we compute the maximal length $L(m)$ of a periodic factor of period m, at least in the more interesting case $L(m) \geqslant 2$. To this aim we give and use a combinatorial presentation of the well-known three distance theorem (Theorem 2). The method, inspired by [6], makes use of a bijection [4] between the factors of length m and a partition of the circle into $m+1$ intervals. The expressions for $L(m)$ are obtained in terms of denominators of the convergents in the representation of the "slope" of s by a continued fraction (Theorem 4).

As this result gives a precise expression for the maximal exponent $L(m)/m$ in function of m it allows to recover all the aformentioned results. This will be precised in Section 4.2 where some further properties of maximal fractional powers and examples are also given.

2. Preliminaries

2.1. Sturmian words

Let $A=\{0,1\}$ be a two letter alphabet. *Words* over A are finite (possibly empty) sequences $u=u(1)u(2)\cdots u(m)$, $m \in \mathbb{N}_+$, (the set of positive integers), $u(i) \in A$. The *length* of u is $|u|=m$. Similarly, *infinite words* are infinite sequences $s=s(1)(2)\cdots s(i)\ldots$, $i \in \mathbb{N}_+$, $s(i) \in A$. A word u is a *factor* of a finite or infinite word t if $t=t'ut''$, where t' is a word and t'' is a finite or infinite word according to the case. When t' is the empty word, u is a *prefix* of t. Also if $u=t(i)t(i+1)\cdots t(j)$ we write $u=t(i,j)$. The set of the factors of t is denoted by $F(t)$.

Let $u=u(1)u(2)\cdots u(n)$ be a word of length n and let $m \in \mathbb{N}_+$, $m \leqslant n$. Then u is periodic with period m if $u(i+m)=u(i)$ for $1 \leqslant i \leqslant n-m$. In this case we can equivalently say that u is a *fractional power* of $v=u(1)u(2)\cdots u(m)$, with *exponent* $e=n/m$, and write $u=v^e$. When e is an integer, we find the classical definition of powers.

Let $\alpha \in [0,1]$ be an irrational number. Then the *standard Sturmian word* of *slope* α (also called the *characteristic sequence of* α) is the infinite word s given by

$$s(n)=\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor, \quad n \in \mathbb{N}_+$$

or equivalently by

$$s(n)=0 \quad \text{if } \{n\alpha\}\in[0,1-\alpha[$$

$$s(n)=1 \quad \text{if } \{n\alpha\}\in[1-\alpha,1[$$

where as usual $\{x\}=x-\lfloor x\rfloor$ denotes the fractional part of the real x.

An infinite word t is *Sturmian* if $F(t)=F(s)$ for some standard Sturmian word s. Among the many equivalent algebraic or combinatorial definitions of Sturmian words, let us recall the following one which is of concern for our purpose. An infinite word is Sturmian if and only if for all $n\in\mathbb{N}$ it has exactly $n+1$ factors of length n.

Hereafter, as we are interested only in factors, we limit ourselves to the consideration of standard Sturmian words.

2.2. Simple continued fractions [2, Chap. X]

Let $\alpha\in[0,1]$ be an irrational number and $[a_0,a_1,\ldots]$ be its representation by a simple continued fraction. As $\alpha\in[0,1]$, we have $a_0=0$. Also $a_1,a_2,\ldots$ are positive integers. The convergents are fractions p_n/q_n, $n\in\mathbb{N}$ with p_n,q_n defined by the recurrence relations

$$p_0=a_0,\quad p_1=a_1a_0+1,\quad p_{n+1}=a_{n+1}p_n+p_{n-1},\qquad (n\geqslant 1)$$

$$q_0=1,\quad q_1=a_1,\quad q_{n+1}=a_{n+1}q_n+q_{n-1},\qquad (n\geqslant 1)$$

As customary we extend the general formulas to $n=0$ by putting $p_{-1}=1,\ q_{-1}=0$. By Theorem 164 of [2] we have, for n even for instance

$$1>q_n\alpha-p_n>q_{n+2}\alpha-p_{n+2}>0 \tag{1}$$

whence, as $q_{n+2}=q_n+a_{n+2}q_{n+1}$ and $p_{n+2}=p_n+a_{n+2}p_{n+1}$,

$$\begin{aligned}1&>q_n\alpha-p_n>(q_n+q_{n+1})\alpha-(p_n+p_{n+1})>\cdots>((q_n+(a_{n+2}-1)q_{n+1})\alpha\\&\quad-(p_n+(a_{n+2}-1)p_{n+1})>q_{n+2}\alpha-p_{n+2}>0\end{aligned} \tag{2}$$

It also follows from (1) and from Theorem 164 of [2]

$$q_{n+2}\alpha-p_{n+2}+(-q_{n+1}\alpha+p_{n+1})>-q_{n+1}\alpha+p_{n+1}>0>q_{n+2}\alpha-p_{n+2},$$

that is

$$\begin{aligned}&(q_n+(a_{n+2}-1)q_{n+1})\alpha-(p_n+(a_{n+2}-1)p_{n+1})>-q_{n+1}\alpha+p_{n+1}\\&\quad>(q_n+(a_{n+2}+1)q_{n+1})\alpha-(p_n+(a_{n+2}+1)p_{n+1})>0.\end{aligned} \tag{3}$$

Using (2) and (3) and observing that positive members belong to $]0,1[$ we get, for even $n\geqslant 2$

$$\begin{aligned}1&>\{q_n\alpha\}>\{(q_n+q_{n+1})\alpha\}>\{(q_n+2q_{n+1})\alpha\}\\&>\cdots>\{(q_n+(a_{n+2}-1)q_{n+1})\alpha\}>\{-q_{n+1}\alpha\}>0\end{aligned} \tag{4}$$

In the same way, for odd $n \geqslant 1$ we have

$$1 > \{-q_n\alpha\} > \{-(q_n + q_{n+1})\alpha\} > \{-(q_n + 2q_{n+1})\alpha\}$$
$$> \cdots > \{-(q_n + (a_{n+2} - 1)q_{n+1})\alpha\} > \{q_{n+1}\alpha\} > 0. \tag{5}$$

3. Factors of Sturmian words

Let s be a Sturmian word. Then for any $m \in \mathbb{N}_+$ there exists a well-known bijection between $F_m(s) = A^m \cap F(s)$ and some partition of the circle of length 1 into $m+1$ intervals [8, p. 26, 4, 6]. As in [6] instead of considering the directed circle of length 1 we will consider the interval $[0,1[$ of $\mathbb{R}$ with some conventions.

(a) Let $\lambda, \mu \in [0,1[$, then the (*generalized*) interval $[\lambda,\mu[$ is the customary interval $[\lambda,\mu[$ if $0 \leqslant \lambda < \mu < 1$ and is $[\lambda,1[\cup [0,\mu[$ if $0 \leqslant \mu < \lambda < 1$. Also $[\lambda,\lambda[$ is the empty set. Accordingly, the *amplitude* $|[\lambda,\mu[|$ of $[\lambda,\mu[$ is $\{\mu - \lambda\}$ that is $\mu - \lambda$ if $\lambda \leqslant \mu$ and $\mu + 1 - \lambda$ if $\mu < \lambda$.

(b) If $\mu, \lambda, v \in [0,1[$ and $\mu \in [\lambda, v[$ we write $\lambda \prec \mu \prec v$. This means that when starting from λ in the positive direction on the circle of length 1 we meet μ before v. More generally $\lambda \prec \mu_1 \prec \mu_2 \prec \cdots \prec \mu_n \prec v$ means that $[\lambda,\mu_1[, \ldots, [\mu_n, v[$ are disjoint and $|[\lambda, v[| = |[\lambda,\mu_1[| + \cdots + |[\mu_n, v[|$. We will often make use of evident properties of “translations modulo 1”. For instance $\mu \in [\lambda, v[\Leftrightarrow \{\mu + \rho\} \in [\{\lambda + \rho\}, \{v + \rho\}[$ for $\lambda, \mu, v \in [0,1[$ and any real ρ.

Now, considering the standard Sturmian word $s = s_\alpha$ with slope α and $m \in \mathbb{N}_+$, denote by c_i, $0 \leqslant i \leqslant m$ the $\{-j\alpha\}$, $0 \leqslant j \leqslant m$ in increasing order: $0 = c_0 < c_1 < \cdots < c_m < 1$ and put $I_i = [c_i, c_{i+1}[$, $0 \leqslant i < m$, and $I_m = [c_m, 0[$. We recall

Theorem 1 (Knuth [4], Mignosi [6]). *There exists a bijection between $F_m(s)$ and the set of the I_k. More precisely the value of $s(n, n+m-1)$, $n \in \mathbb{N}_+$ is determined by the I_k containing $\{n\alpha\}$.*

Consequently we are led to a study of the I_k. Theorem 2 hereafter is essentially the 3-distance theorem [11, 12, 1], but with a presentation suited to our purpose.

Let $[a_0, a_1, a_2, \ldots]$ be the simple continued fraction representing α with notations for the convergents as in the Preliminaries (here $a_0 = 0$ as $0 < \alpha < 1$). For $m \in \mathbb{N}_+$ there exists a unique $i \in \mathbb{N}$ such that $q_{i-1} + q_i \leqslant m < q_i + q_{i+1}$. We define $b, r \in \mathbb{N}$ by putting

$$m = q_{i-1} + bq_i + r, \quad 1 \leqslant b \leqslant a_{i+1}, \quad 0 \leqslant r < q_i \tag{6}$$

Define a circular permutation φ_m of $\{0, 1, \ldots, m\}$ by
if $0 \leqslant x \leqslant m - q_i$ then $\varphi_m(x) = x + q_i$
if $m - q_i < x < m - r$ then $\varphi_m(x) = x + q_i + r - m$
if $m - r \leqslant x \leqslant m$ then $\varphi_m(x) = x + r - m$.

Putting $\psi_m = \varphi_m^{-1}$ we also have:
if $0 \leqslant x \leqslant r$ then $\psi_m(x) = x + m - r$
if $r < x < q_i$ then $\psi_m(x) = x + m - r - q_i$
if $q_i \leqslant x \leqslant m$ then $\psi_m(x) = x - q_i$.

Lastly for any $n, n' \in \mathbb{Z}$ we write for shortness $\langle n, n' \rangle = [\{-n\alpha\}, \{-n'\alpha\}[$.

All those notations and conventions will be valuable throughout this paper.

Theorem 2. *Let $m \in \mathbb{N}_+$, with notations as above. Then the corresponding I_k are given by*

$$I_k = \langle \varphi_m^k(0), \varphi_m^{k+1}(0) \rangle \text{ if } i \text{ is odd}$$

$$(\text{resp. } I_k = \langle \psi_m^k(0), \psi_m^{k+1}(0) \rangle \text{ if } i \text{ is even})$$

The amplitudes of these intervals have at most three values, $l_1, l_2 = l_1 + l_3, l_3$ for, respectively, $m - q_i - 1$, $q_i - r - 1$ and $r + 1$ intervals. In particular there is no interval of amplitude l_2 if $r = q_i - 1$. These amplitudes are $l_1 = \{-q_i\alpha\}$, $l_2 = \{(m - q_i - r)\alpha\}$, $l_3 = \{(m - r)\alpha\}$ if i is odd (resp. $l_1 = \{q_i\alpha\}$, $l_2 = \{-(m - q_i - r)\alpha\}$, $l_3 = \{-(m - r)\alpha\}$ if i is even).

Lastly, $l_1 < l_3$ if $b < a_{i+1}$, i.e. $m < q_{i+1}$ and $l_3 < l_1$ if $b = a_{i+1}$, i.e. $q_{i+1} \leqslant m$.

Proof. We proceed by induction on m. When $m = 1$, as $q_{-1} + q_0 = 1$ we have $i = 0$, $b = 1$, $\psi_1(0) = 1$, $\psi_1(1) = 0$, $I_0 = \{0, -\alpha\}$, $I_1 = \{-\alpha, 0\}$ hence $I_0 = \langle \psi_1^0(0), \psi_1(0) \rangle$, $I_1 = \langle \psi_1(0), \psi_1^2(0) \rangle$, $l_1 = \{\alpha\}$, $l_3 = \{-\alpha\}$. At last if $b < a_1$ we have $a_1 > 1$ whence $\alpha < 1/2$ whence $l_1 < l_3$, and if $b = a_1$, then $a_1 = 1$ whence $\alpha > 1/2$ whence $l_3 < l_1$. So the theorem is verified for $m = 1$. Now supposing the theorem true for some $m \in \mathbb{N}_+$, let us prove it for $m' = m + 1$. With the same notations as previously, let $m = q_{i-1} + bq_i + r$. Suppose for instance i is odd (the case i even is similar). For simplicity of notation we write now φ instead of φ_m. The intervals of $[0, 1[$ corresponding to m' are the same as those, I_k, $k \in \{0, 1, \ldots, m\}$, corresponding to m, except that the one containing $\{-m'\alpha\}$ is divided in two parts. As $\varphi(m - q_i) = m$ and $\varphi(m) = r$ we have

$$\{-m\alpha\} \in [(-m - q_i)\alpha, -m\alpha[\cup [-m\alpha, -r\alpha[= \langle m - q_i, r \rangle.$$

By translation modulo 1 of value $-\alpha$ we get

$$\{-m'\alpha\} \in J = \langle m' - q_i, r' \rangle$$

with $r' = r + 1$.

Consider three cases:

(a) *Case* 1: $r < q_i - 1$. Then $m' = q_{i-1} + bq_i + r'$, $r' < q_i$. In this case, $\varphi(m' - q_i) = r'$ whence J is one of the I_k, I_h say, and has amplitude l_2. Put $J_1 = \langle m' - q_i, m' \rangle$, $J_2 = \langle m', r' \rangle$ and define a permutation φ' of $\{0, 1, \ldots, m'\}$ by $\varphi'(x) = \varphi(x)$ for $x \neq m' - q_i$, $\varphi'(m' - q_i) = m'$, $\varphi'(m') = r'$. An easy verification shows that φ' is $\varphi_{m'}$, and that the intervals, I'_t say, corresponding to m' are $\langle \varphi_{m'}^t(0), \varphi_{m'}^{t+1}(0) \rangle$, $0 \leqslant t \leqslant m'$ and are related to the I_k by $I'_t = I_t$, $0 \leqslant t < h$, $I'_h = J_1$, $I'_{h+1} = J_2$, $I'_t = I_{t-1}$, $h + 1 < t \leqslant m'$.

As J_1 and J_2 have amplitude l_1 and l_3, respectively, we gain one interval of each amplitude l_1, l_3 and loss one interval of amplitude l_2 when passing from m to m'. Lastly, as m and m' have the same value for b, the condition $l_1 < l_3 \Leftrightarrow b < a_{i+1}$ is satisfied for m'.

(b) *Case* 2: $r = q_i - 1$ and $b < a_{i+1}$. We have now $m' = q_{i-1} + b'q_i \leqslant q_{i+1} < q_i + q_{i+1}$, with $b' = b + 1$. In this case $r' = r + 1 = q_i$ and $\varphi(m' - q_i) = \varphi(q_{i-1} + bq_i) = 0$ and $\varphi(0) = r'$. So $J = J_1 \cup J_2$ where $J_1 = \langle m' - q_i, 0\rangle$ and $J_2 = \langle 0, q_i\rangle$ are two intervals I_k of amplitude, respectively, l_3 and l_1. As $b < a_{i+1}$ and the theorem is true for m, we have $l_1 < l_3$. As $[\{-(m' - q_i)\alpha\}, \{-m'\alpha\}[= \langle m' - q_i, m'\rangle$ has amplitude l_1 it follows $\{-m'\alpha\} \in J_1$. Define a permutation φ' of $\{0, 1, \ldots, m'\}$ by $\varphi'(x) = \varphi(x)$ for $x \neq m' - q_i$, $\varphi'(m' - q_i) = m'$, $\varphi'(m') = 0$. Then it is easily verified that $\varphi' = \varphi_{m+1} = \varphi_{m'}$, and that the intervals I'_t corresponding to m' are the $\langle \varphi^t_{m'}(0), \varphi^{t+1}_{m'}(0)\rangle$.

The amplitudes of the I'_t are $l'_1 = l_1$, $l'_2 = l_3 = l'_1 + l'_3$, $l'_3 = l_3 - l_1 = \{(q_{i-1} + b'q_i)\alpha\}$. These amplitudes and the number of occurrences of each of them satisfy the theorem as it is easily verified. It remains to compare l'_1 and l'_3. It follows from formula (4) in the Preliminaries that if $b' < a_{i+1}$ then $\{(q_{i-1} + b'q_i)\alpha\} > \{-q_i\alpha\}$, that is $l'_3 > l'_1$. Also if $b' = a_{i+1}$ then by formula (5) $l'_3 = \{q_{i+1}\alpha\} < \{-q_i\alpha\} = l'_1$. So this part of the theorem is verified for m'.

(c) *Case* 3: $r = q_i - 1$ and $b = a_{i+1}$. Then $m' = q_{i-1} + a_{i+1}q_i + q_i = q_i + q_{i+1}$, whence $q_{i'-1} + q_{i'} \leqslant m' < q_{i'} + q_{i'+1}$ with $i' = i+1$. As in the second case, we have $\{-m'\alpha\} \in J = J_1 \cup J_2$ with J_1 and J_2 being intervals I_k of respective amplitude l_3 and l_1. But now $l_3 < l_1$ whence $\{-m'\alpha\} \in J_2 = \langle 0, q_i\rangle$. We define a permutation φ' by $\varphi'(x) = \varphi^{-1}(x)$ for $0 < x \leqslant m$, $\varphi'(0) = m'$, $\varphi'(m') = q_i$. We easily verify that φ' is $\psi_{m'}$, and that the intervals I'_j, $j = 0, 1, \ldots, m'$, corresponding to m' are the $\langle \psi^j_{m'}(0), \psi^{j+1}_{m'}(0)\rangle$.

The amplitudes of the I'_j are $l'_1 = l_3$, $l'_2 = l_1 = l'_1 + l'_3$, $l'_3 = l_1 - l_3$ and their numbers of occurrences are, respectively, $q_i + 1$, $q_{i+1} - 1$ and 1, that is $m' - q_{i'-1} + 1$, $q_{i'} - 1$ and $0+1$ as stated for m' by the theorem. It remains to compare l'_1 and l'_3. As $m' = q_i + q_{i+1}$ we have $l'_1 = \{q_{i+1}\alpha\}$ and $l'_3 = \{-(q_i + q_{i+1})\alpha\}$. Also $q_{i+2} = q_i + a_{i+2}q_{i+1}$. So if $a_{i+2} > 1$ it follows from formula (5) that $l'_3 > l'_1$, while if $a_{i+2} = 1$, by formula (4), $l'_1 < l'_3$. These two cases correspond respectively to $m' < q_{i+2}$ and $m' = q_{i+2}$. $\square$

This theorem gives a comparison between l_1 and l_3, however in view of Section 4, a more precise comparison is wanted.

Theorem 3. (i) *With notations as previously*,

$$(a_{i+1} - b)l_1 < l_3 < (a_{i+1} + 1 - b)l_1. \tag{7}$$

(ii) *If* $b = a_{i+1}$ (*or equivalently* $m \geqslant q_{i+1}$) *then*

$$a_{i+2}l_3 < l_1 < (a_{i+2} + 1)l_3. \tag{8}$$

Proof. With i odd for instance, we have by the Preliminaries, formula (2), $\{q_{i-1}\alpha\} > h\{-q_i\alpha\} > 0$ for $1 \leqslant h \leqslant a_{i+1}$. It follows $\{q_{i-1}\alpha\} - b\{-q_i\alpha\} = \{(q_{i-1} + bq_i)\alpha\}$ whence

$\{q_{i-1}\alpha\} = l_3 + bl_1$. By the Preliminaries, formulas (2, 3), we have $a_{i+1}l_1 < \{q_{i-1}\alpha\} < (a_{i+1}+1)l_1$ and (7) follows.

Now if $b = a_{i+1}$ we have $q_{i-1} + bq_i = q_{i+1}$ whence $l_3 = \{q_{i+1}\alpha\}$. Using the dual for n odd of formulas (2, 3) of Preliminaries we get (8). $\square$

4. Periodic factors of Sturmian words

4.1. Calculation of $L(m)$

Given the standard Sturmian word s of slope α we define, for $m \in \mathbb{N}_+$, $L(m)$ by $L(m) = \ Sup\{|w|; w \in F(s)$ and w has period $m\}$.

We will compute $L(m)$, but as the case $L(m) < 2m$ leads to complicated formulas, we limit ourselves here to the case $L(m) \geqslant 2m$.

The notations are those of Section 3, in particular (6) holds and the I_k are those of Theorem 2. For simplicity, we write φ, ψ instead of φ_m, ψ_m.

Part A of Theorem 4 gives the cases where $L(m) \geqslant 2m$ and Part B gives the value of $L(m)$. There are five cases (and even seven as cases (iii) and (v) contain a disjunction in their formulation). This is due to the intervention of some relative values of the parameters. For instance when $r = q_i - 1$ there is no interval I_k of size l_2. However only cases (i) and (ii) are general, the other ones imply $i \in \{0, 1, 2\}$ hence concern small values of m. An acceptable assumption $\alpha < 1/2$ or $\alpha > 1/2$ could also eliminate case (iv) or (v) but here we have preferred uniformity.

In order to calculate $L(m)$ we need a lemma.

Lemma 1. *Let λ, μ, v_0, ρ be real numbers and $h \in \mathbb{N}_+$ such that putting $v_j = v_0 + j\rho$, $0 \leqslant j \leqslant h$ we have $\lambda, \mu \in [0, 1[$ and λ, μ and all $\{v_j\}$ are different and $\{v_j\} \in [\lambda, \mu[$ for $0 \leqslant j \leqslant h$. Then*

(a) *if $\{\mu - \lambda\} + \{\rho\} \leqslant 1$ then $\lambda \prec v_0 \prec v_1 \prec \cdots \prec v_h \prec \mu$ and $h\{\rho\} < \{\mu - \lambda\}$*

(b) *if $\{\mu - \lambda\} + \{-\rho\} \leqslant 1$ then $\lambda \prec v_h \prec v_{h-1} \prec \cdots \prec v_0 \prec \mu$ and $h\{-\rho\} < \{\mu - \lambda\}$.*

Proof. Consider case (a) for instance. Without loss of generality we can suppose $\lambda = 0$ because this amounts to replacing $[\lambda, \mu[$ by $[0, \{\mu - \lambda\}[$ and $\{v_j\}$ by $\{v_j - \lambda\}$. We have then to show that $0 < \{v_0\} < \{v_1\} < \cdots < \{v_h\} < \mu < 1$.

If this is false we have, for some $j \in [1, h]$, $\{v_{j-1}\} > \{v_j\}$, that is $\{v_0 + (j-1)\rho\} > \{v_0 + j\rho\}$. It follows $\{v_0 + j\rho\} = \{v_0 + (j-1)\rho\} + \{\rho\} - 1$ whence $\{v_0 + (j-1)\rho\} + \{\rho\} \geqslant 1$. As $\{v_0 + (j-1)\rho\} < \mu$ we get $\mu + \rho > 1$ contrarily to the hypothesis. $\square$

Theorem 4. *Let s be a standard Sturmian word with slope $\alpha = [a_0 = 0, a_1, a_2, \ldots]$. For any $m \in \mathbb{N}_+$ define integers i, r, b by*

$$q_{i-1} + q_i \leqslant m = q_{i-1} + bq_i + r < q_i + q_{i+1},$$

with $1 \leqslant b \leqslant a_{i+1}$, $0 \leqslant r < q_i$, where $q_{-1} = 0$ and $q_0 = 1$, q_1, q_2, ... are the denominators of the convergents of α.

Let $L(m)$ be the maximal length of the factors of s having period m. Then
(A) *We have $L(m) \geqslant 2m$ if and only if one of the following conditions is satisfied*
(i) $0 = r \neq q_i - q_{i-1}$ *and* $q_i > 1$
(ii) $0 < r = q_i - q_{i-1}$ *and* $q_{i-1} > 1$ *and* $2b \leqslant a_{i+1}$
(iii) $0 = r \neq q_i - q_{i-1}$ *and* $q_i = 1$ *and either* $b = a_{i+1}$ *or* $2b \leqslant a_{i+1}$
(iv) $0 < r = q_i - q_{i-1}$ *and* $q_{i-1} = 1$ *and* $2b < a_{i+1}$
(v) $0 = r = q_i - q_{i-1}$ *and either* $b = a_{i+1}$ *or* $2b < a_{i+1}$.
(B). *When $L(m) \geqslant 2m$ its value is as follows where the cases are those of Part* A.
(i) $L(m) = 2m + q_i - 2$ *if* $b < a_{i+1}$ *and* $L(m) = (2 + a_{i+2})m + q_i - 2 = q_{i+2} + 2q_{i+1} - 2$ *if* $b = a_{i+1}$
(ii) $L(m) = q_{i+1} + 2q_i - 2$
(iii) $L(m) = q_2 + 2q_1 - 2$ *if* $b = a_1$ *and* $L(m) = q_1$ *if* $2b \leqslant a_1$
(iv) $L(m) = q_{i+1} + 2q_i - 2$
(v) $L(m) = q_3 + 2q_2 - 2$ *if* $b = a_2$ *and* $L(m) = q_2$ *if* $2b < a_2$.

Proof of Part A. Clearly, $L(m) \geqslant 2m$ if and only if $s(n, n+m-1) = s(n+m, n+2m-1)$ for some $n \in \mathbb{N}_+$, that is if $\{n\alpha\}$ and $\{(n+m)\alpha\}$ belong to the same interval I_k. As the sequence $\{j\alpha\}$, $j \in \mathbb{N}_+$ is dense in $[0, 1[$ this will occur if and only if $\{m\alpha\} < M$ or $\{-m\alpha\} < M$, where $M = \operatorname{Sup}\{|I_k|; 0 \leqslant k \leqslant m\}$. As $\{m\alpha\} = |\langle m, 0\rangle|$ and $\{-m\alpha\} = |\langle 0, m\rangle|$ we will study these two intervals. Both are union of successive I_k. More precisely, with i odd for instance, define t in $[0, m]$ by $m = \varphi^t(0)$. Then $\langle 0, m\rangle = I_0 \cup I_1 \cup \cdots \cup I_{t-1}$ and $\langle m, 0\rangle = I_t \cup I_{t+1} \cup \cdots \cup I_m$. Hence $|\langle 0, m\rangle| \geqslant l_1 + l_3$, unless either $r = q_i - q_{i-1}$, which gives $m = (b+1)q_i$ and $|\langle 0, m\rangle| = (b+1)l_1$, or $r = q_{i-1} = 0$ which gives $i = 0$, $m = b$, $|\langle 0, m\rangle| = ml_1$. In the same way $|\langle m, 0\rangle| \geqslant l_1 + l_3$, unless $r = 0$ which gives $|\langle m, 0\rangle| = l_3$. Also observing that $M = l_2$ if $r < q_i - 1$ and that $M = \operatorname{Sup}\{l_1, l_3\}$ if $r = q_i - 1$, we are led to the cases of the theorem.

If $0 = r \neq q_i - q_{i-1}$ and $q_i > 1$ then $M = l_2 > l_3$, so $L(m) \geqslant 2$. This is case i).

If $0 < r = q_i - q_{i-1}$ and $q_{i-1} > 1$ then $L(m) \geqslant 2m$ if and only if $(b+1)l_1 < M = l_2 = l_1 + l_3$. Using formula (7) this gives $2b \leqslant a_{i+1}$, this is case (ii).

If $0 = r \neq q_i - q_{i-1}$ and $q_i = 1$ then $i = 0$ and $q_{i-1} = 0$ and $M = \operatorname{Sup}\{l_1, l_3\}$. So $L(m) \geqslant 2m$ if and only if either $l_3 < l_1$, that is $b = a_{i+1}$, or $bl_1 < l_3$, that is $2b \leqslant a_{i+1}$ (case iii)).

If $0 < r = q_i - q_{i-1}$ and $q_{i-1} = 1$ then in the same way $L(m) \geqslant 2m$ if and only if $2b < a_{i+1}$ (case iv)).

If $0 = r = q_i - q_{i-1}$ then $i = 1$, $q_i = q_{i-1} = 1$, $M = \operatorname{Sup}\{l_1, l_3\}$. So $L(m) \geqslant 2m$ if and only if either $l_3 < l_1$, that is $b = a_{i+1}$, or $(b+1)l_1 < l_3$, that is $2b < a_{i+1}$ (case v)).

Proof of Part B. Let $v = s(n, n + L(m) - 1)$ be a factor of s of length $L(m)$ and period m. Put $u = s(n, n + m - 1)$ and $L(m) = em + z$, $e, z \in \mathbb{N}$ and $0 \leqslant z < m$. Then $v = u^e w$ with w a prefix of u of length z. Consequently there exists an I_k containing $\{n\alpha\}$, $\{n\alpha + m\alpha\}$, …, $\{n\alpha + (e-1)m\alpha\}$ but containing neither $\{n\alpha - m\alpha\}$ nor $\{n\alpha + em\alpha\}$ as $u^{e+1} \notin F(s)$.

Now, for any $y\in\mathbb{Z}$, let $S(y)=\{\{n\alpha+y\alpha+jm\alpha\};\ j=0,1,\dots,e-1\}$. If $0\leqslant y\leqslant z$, then $s(n+y,n+y+em-1)$ is an eth power of some conjugate of u whence $S(y)\subset I(y)$, where $I(y)$ is one of the I_k. On the contrary $S(-1)$ and $S(z+1)$ are not included in any I_k. If $I(0)=[\lambda,\mu[$, put for simplicity $T(y)=[\{\lambda+y\alpha\},\{\mu+y\alpha\}[$ for any $y\in\mathbb{Z}$. It follows $S(y)\subset T(y)$. We will consider the five cases corresponding to $L(m)\geqslant 2$. In each of them we will make use of Lemma 1 applied to $S(0)$ and $I(0)$. The verification that the hypotheses of this lemma are satisfied becomes trivial in view of the following remarks.

Let $\Delta=\ Inf\{\{m\alpha\},\{-m\alpha\}\}$ and as previously $M=\ Sup\{|I_k|;0\leqslant k\leqslant m\}$. Then the hypothesis $\{\mu-\lambda\}+\{\rho\}\leqslant 1$ or $\{\mu-\lambda\}+\{-\rho\}\leqslant 1$ will be satisfied if $M+\Delta\leqslant 1$, hence in particular if $M<1/2$. By Theorem 2 we have $(m-q_i+1)l_1+(q_i-r-1)l_2+(r+1)l_3=1$. If $q_i\neq r+1$ it follows easily $M=l_2<1/2$. If $q_i=r+1$ then $M=\ Sup\{l_1,l_3\}$ and $(m-r)l_1+q_il_3=1$. Suppose firstly that $M=l_3$. Then if $M>1/2$ we must have $q_i=1$ whence $r=0$, $\Delta=ml_1$ and $M+\Delta=1$. Secondly suppose $M=l_1$ Then if $M>1/2$ we must have $m-r=1$ whence $m=q_i$ whence $q_{i-1}=0$, $i=0$, $m=q_0=1$ and $\Delta=l_3$ whence $M+\Delta=1$.

The following lemma will simplify the proof of some cases.

Lemma 2. *If $I(0)>l_1$ and $q_i|m$ then $L(m)=L(q_i)$.*

Proof. With $v=s(n,n+L(m)-1)$ and $I(0)=[\lambda,\mu[$ as previously we have for instance $\lambda\prec\{n\alpha\}\prec\{(n+m)\alpha\}\prec\mu$ (the case $\lambda\prec\{(n+m)\alpha\}\prec\{n\alpha\}\prec\mu$ is similar).

As $\{j\alpha;j\in\mathbb{N}_+\}$ is dense in $[0,1[$ we can choose n in such a way that $\mu-\{(n+m)\alpha\}$ is arbitrarily small positive. As $l_1=\{-q_i\alpha\}<|I(0)|$ we get that $\{(n+m-q_i)\alpha\}\in I(0)$ whence $s(n+m-q_i,n+2m-q_i-1)=s(n+m,n+2m-1)=u$ So u has period q_i. Moreover as $q_i|m$, v also has period q_i whence $L(m)\leqslant L(q_i)$. As trivially any word with period q_i also has period m, $L(m)\geqslant L(q_i)$ and the result follows. □

Now successively consider cases (i) to (v) of Part A.

(i) Case $0=r\neq q_i-q_{i-1}$ and $q_i>1$. First suppose i is odd. We have $I(0)=\langle x,\varphi(x)\rangle$ for some $x\in\{0,1,\dots,m\}$. Then $\{m\alpha\}=|\langle m,0\rangle|=l_3$. As $S(0)\subset I(0)$ we have $S(-1)\subset T(-1)=\langle x+1,\varphi(x)+1\rangle$. As this one is not an I_k we get $x=m$ or $\varphi(x)=m$ or $0\leqslant x\leqslant m$ and $\varphi(x)+1\neq\varphi(x+1)$. This gives for $I(0)$ the three possibilities $J_1=\langle m,0\rangle$, $J_2=\langle m-q_i,m\rangle$, $J_3=\langle m-1,q_i-1\rangle$. As $|J_1|=l_3$, J_1 is excluded. If $I(0)=J_2$ then $|I(0)|=l_1$ and $T(-1)=\langle m-q_i+1,m+1\rangle$. As $\langle m-q_i+1,1\rangle$ is an I_k of amplitude l_2, $S(-1)\subset T(-1)\subset\langle m-q_i+1,1\rangle$ and this is a contradiction. Consequently $I(0)=\langle m-1,q_i-1\rangle$ and has amplitude l_2.

Observing now that for $0\leqslant y\leqslant q_i-2$, $S(y)\subset T(y)=\langle m-1-y,q_i-1-y\rangle$ which is an I_k we get $z\geqslant q_i-2$. Also $S(q_i-1)\subset\langle m-q_i,0\rangle=\langle m-q_i,m\rangle\cup\langle m,0\rangle$. If $S(q_i-1)$ were included in $\langle m-q_i,m\rangle$ which is an I_k of amplitude l_1 we would have $(e-1)l_3<l_1$ whence $el_3<l_2$ whence s would contain an $(e+1)$th power of some conjugate of u, a contradiction. As $S(q_i-1)$ is not included in $\langle m,0\rangle$ which is an I_k of amplitude l_3, $S(q_i-1)$ is not included in any I_k. Consequently $z=q_i-2$.

If we suppose i even a similar argument gives the same conclusion.

Lastly, using Lemma 1, $e-1=\lfloor l_2/l_3\rfloor=1+\lfloor l_1/l_3\rfloor$ whence by formulas (7) and (8) $e=2$ if $b<a_{i+1}$ and $e=2+a_{i+2}$ if $b=a_{i+1}$, whence the values as claimed for $L(m)=em+z$.

(ii) Case $0<r=q_i-q_{i-1}$ and $q_{i-1}>1$ and $2b\leqslant a_{i+1}$. Suppose i is odd for instance. Now $m=(b+1)q_i$ and $\{-m\alpha\}=|\langle 0,m\rangle|=(b+1)l_1$. Writing as previously that $T(-1)$ is not an I_k we get for $I(0)$ the possible values $J_1=\langle m,r\rangle$, $J_2=\langle m-q_i,m\rangle$, $J_3=\langle m-r-1,q_i-1\rangle$. As $|J_2|=l_1$, J_2 is excluded. If $I(0)=J_1$ then $T(-1)=\langle m+1,r+1\rangle$ has amplitude l_3. As $\langle bq_i+1,r+1\rangle$ is an I_k of amplitude l_2, $S(-1)\subset T(-1)\subset\langle bq_i+1,r+1\rangle$, a contradiction. Consequently $I(0)=\langle m-r-1,q_i-1\rangle$ of amplitude l_2.

As $I(0)>l_1$ and $q_i|m$, by Lemma 2, $L(m)=L(q_i)$. Now putting $q_i=m'$, $i-1=i'$ we have $m'=q_{i'-1}+b'q_{i'}+r'$ with $b'=a_{i'+1}$, $r'=0$.

So we are in case (i) if $q_{i'}>1$ and in case (iii) if $q_{i'}=1$. In case (i) we get $L(q_i)=q_{i'+2}+2q_{i'+1}-2=q_{i+1}+2q_i-2$. In case (iii), noting that $i'=0$, we get $L(q_i)=q_2+2q_1-2=q_{i+1}+2q_i-2$.

(iii) Case $0=r\neq q_i-q_{i-1}$ and $q_i=1$ and $b=a_{i+1}$ or $2b\leqslant a_{i+1}$. This implies $q_{i-1}=0$, $i=0$, $m=b$, $M=\mathit{Sup}\{l_1,l_3\}$, $l_3+ml_1=1$, $\{-m\alpha\}=l_3$, $\{m\alpha\}=bl_1$. As i is even, we have $I(0)=\langle x,\psi(x)\rangle$. The possibility for $I(0)$ are now $J_1=\langle m,m-1\rangle$, $J_2=\langle 0,m\rangle$. Consider two subcases

(a) $b=a_{i+1}=a_1$, whence $l_3<l_1=M$. As J_2 has amplitude l_3, J_2 is excluded. So $I(0)=\langle m,m-1\rangle$ of amplitude l_1. So for $0\leqslant y\leqslant m-1$, $S(y)\subset T(y)=\langle m-y,m-1-y\rangle$ which is an I_k. So $z\geqslant m-1$. Also $S(m)\subset T(m)=\langle 0,-1\rangle\subset\langle 0,m\rangle\cup\langle m,m-1\rangle=I_0\cup I_1$. We have $|I_0|=l_3$ and $|I_1|=l_1$. Clearly, $S(m)\not\subset I_0$. If $S(m)\subset I_1$ we get $S(m)\subset T(m)\cap I_1=\langle m,-1\rangle$. As $|\langle m,-1\rangle|=l_1-l_3$ we get $(e-1)l_3<l_1-l_3$ whence $el_3<l_1$ and then s contains an $(e+1)$th power of some conjugate of u. As this is impossible $S(m)$ is not included in any I_k whence $z=m-1=q_1-1$.

As $e-1=\lfloor l_1/l_3\rfloor=a_2$ we get $L(m)=(a_2+1)q_1+q_1-1=q_2+2q_1-2$. (b) $2b\leqslant a_{i+1}=a_1$ whence $l_1<l_3=M$. As $\{m\alpha\}\geqslant l_1$ and $\{-m\alpha\}>l_1$ the possibility J_1 for $I(0)$ is excluded, so $I(0)=J_2=\langle 0,m\rangle$ of amplitude l_3. As $i=0$, $q_i=1$, so by Lemma 2, $L(m)=L(1)$. Also as $2b\leqslant a_1$, $a_1>1$. Hence as it is well known the maximal power of a letter in s is a_1, so $L(m)=a_1=q_1$.

(iv) Case $0<r=q_i-q_{i-1}$ and $q_{i-1}=1$ and $2b<a_{i+1}$. This implies $i=1$ or $i=2$, $m=(b+1)q_i$, $\{m\alpha\}\geqslant l_1+l_3$, $\{-m\alpha\}=(b+1)l_1$. This case recalls case (ii) but as $r=q_i-1$ there is no I_k of amplitude l_2. We have for $I(0)$ the possibilities $J_1=\langle m,r\rangle$, $J_2=\langle m-q_i,m\rangle$ if i is odd (resp. $J_1=\langle r,m\rangle$, $J_2=\langle m,m-q_i\rangle$ if i is even) so as $|J_2|=l_1<l_3$, $I(0)=J_1$ of amplitude l_3. By Lemma 2 we have then $L(m)=L(q_i)$. Defining $m'=q_i$, $i'=i-1$, b' as above we are for m'

in case (i) if $q_{i'}>1$ which implies $i'=1$

in case (iii) if $q_{i'}=1>q_{i'-1}$ which implies $i'=0$

in case (v) if $q_{i'}=q_{i'-1}=1$ which implies $i'=1$.

Using the formulas for these cases with $b'=a_{i'+1}$ we always get $L(m)=L(q_i)=q_{i+1}+2q_i-2$.

(v) Case $0=r=q_i-q_{i-1}$ and either $b=a_{i+1}$ or $2b<a_{i+1}$. This implies $q_i=q_{i-1}=1$,

$i=1$, $m=1+b$, $\{m\alpha\}=l_3$, $\{-m\alpha\}=(b+1)l_1$ and the possibilities for $I(0)$ are $J_1=\langle m,0\rangle$ of amplitude l_3, $J_2=\langle m-1,m\rangle$ of amplitude l_1. We have two subcases

(a) $b=a_{i+1}$. Then $l_3<l_1$ whence $I(0)=J_2=\langle m-1,m\rangle$, also $m=q_{i+1}$. Reasoning as in subcase a) of case (iii) we obtain $L(m)=(a_{i+2}+1)q_{i+1}+q_{i+1}-1=q_3+2q_2-2$.

(b) $2b<a_{i+1}$ Then $l_1<l_3$ whence $I(0)=\langle m,0\rangle$ of amplitude l_3. By Lemma 2, $L(m)=L(1)$. As $a_1=1$ it is well known that $L(1)=1+a_2=q_2$. □

4.2. Complements and examples

Some known notions used here can be found in [5, Ch. 2] for instance. As previously $L(m)$ is the maximal length of factors having period m. We give two propositions about them.

Proposition 1. *When $L(m)\geqslant 2m$ all factors of s having period m and length $L(m)$ are equal. Their common value v is a palindrome and $v(1,L(m)-m)$ is left special.*

Proof. Let $s(n,n+L(m)-1)=v$ have period m. Then $s(n-1)\neq s(n+m-1)$ so $s(n,n+L(m)-m-1)$ is left special and is unique. So v is unique. As $\tilde{v}$ occurs in s, $v=\tilde{v}$, v is a palindrome. □

Now define $\Lambda(m)$ to be the length of the longest prefix of s having period m.

Proposition 2. (a) *if $u=s(1,m)$ is a primitive word then $\Lambda(m)=L(m)-m$ and $s(1,\Lambda(m))$ is a palindrome.*

(b) *if $u=g^e$, $e\in\mathbb{N}_+$, g primitive, then $L(m)=L(m/e)$ and $\Lambda(m)=\Lambda(m/e)=L(m)-m/e$.*

Proof. (a) Let $w=s(1,L(m)-m)$. As s is standard Sturmian w is a prefix and also a suffix of $v=s(n,n+L(m)-1)$ of Proposition 1. So $\Lambda(m)\geqslant L(m)-m$. Now let w_1 be the longest left special suffix of v. Then $w_1=u_1w$ for some u_1, $0\leqslant|u_1|<m$. If $|u_1|>0$, as both w and w_1 are prefixes of s we get that w_1 and also v have period $|u_1|$. So by the Theorem of Fine and Wilf v has period $p=(m,|u_1|)$ and u is an integral power of a shorter word, a contradiction. So w is the longest left special suffix of v.

Now let $s(t,t+\Lambda(m)-1)$ be an occurrence in s of $s(1,\Lambda(m))$ such that $s(t-1)=s(t+m-1)=u(m)$. As $s(t-1,t+L(m)-m-1)$ is not special, $s(t-2)=s(t+m-2)=u(m-1)$. Continuing this way we see that $s(t,t+\Lambda(m)-1)$ is preceeded by u. So $|s(t-m,t+\Lambda(m)-1)|\leqslant L(m)$ whence $\Lambda(m)=L(m)-m$.

(b) If $u=g^e$, g primitive, $L(m)=L(m/e)$ as in the proof of Lemma 2 and similarly $\Lambda(m)=\Lambda(m/e)$ whence by Part (a) $\Lambda(m)=L(m)-m/e$.

Remark. $s(1,\Lambda(m))$ is a palindrome prefix of s. The structure of such words is known. Let $s_n=L(1,q_n)$ and $h_n=L(1,q_n-2)$ with the q_n and a_n as previously. Then ignoring the shortest ones for simplicity, the palindrome prefixes of s are $s_n^x h_{n-1}$, $n\geqslant 2$, $1\leqslant x\leqslant a_{n+1}$. This word has period q_n and length $q_{n+1}-2$ for $x=a_{n+1}$. So $\Lambda(q_n)\geqslant$

$q_{n+1}-2$. We are not far from the result for case (i) of Theorem 4. It is our opinion that, with some work, a substantial part of Theorem 4 could be deduced this way.

Now let $\xi(m)=L(m)/m$ be the maximal fractional exponent corresponding to period m. Assume for simplicity $\alpha<1/2$ i.e. $a_1>1$. Then

Proposition 3. *The maximum of* $\xi(m)$ *when* $q_{i-1}+q_i\leqslant m<q_i+q_{i+1}$, $i\geqslant 1$, *is* $\xi(q_{i+1})=a_{i+2}+2+(q_i-2)/q_{i+1}$.

Proof. We are in case (i) of Theorem 4. If $b<a_{i+1}$ then $\xi(m)=2+(q_i-2)/m<3$. If $b=a_{i+1}$ then $m=q_{i+1}$ and $L(m)=q_{i+2}+2q_{i+1}-2=(a_{i+1}+2)q_{i+1}+q_i-2$, whence the result. □

This allows to find again all known results about maximal exponents. When the partial quotients a_n are bounded, the exponents are bounded [6] and the maximum integer exponent is $2+\ sup\{a_1-2,a_2,a_3,\ldots\}$.

When the law of the a_n is suitable it is possible to calculate, using (9) hereafter, bounds and limits for the exponents. In particular let $\alpha=[0,a_1,a_2,\ldots,a_h,\overline{a_{h+1},\ldots a_{h+p}}]$ be a quadratic irrational. Recall the general formula (easily deducible from $q_n/q_{n-1}=a_n+q_{n-2}/q_{n-1}$)

$$q_n/q_{n-1}=[a_n,a_{n-1},\ldots,a_2,a_1] \tag{9}$$

Let

$$\lambda_t=[\overline{a_{t-1},a_{t-2},\ldots,a_{h+1},a_{h+p},\ldots,a_t}]$$

for $t\in[h+1,h+p]$.

Then

$$\lim_{x\to\infty}((q_t+xp-1)/(q_t+xp-2))=\lambda_t.$$

For $n>h$, $\lfloor\xi(q_{n-1})\rfloor$ is maximal when $a_n=\ sup\{a_{h+1},\ldots a_{h+p}\}=M$.

It follows that the greatest point of accumulation of the $\xi(m)$ is

$$\Omega=M+2+max\{\lambda_t^{-1};a_t=M,t\in[a_{h+1},\ldots,a_{h+p}]\}$$

It can also be shown that $\xi(m)<\Omega$ except possibly for small values of m. This is the result of [14] in a slightly more general context (s is only the morphic image of a fixed point).

Example 1. With $\alpha=(0,2,\bar{1})=(3-\sqrt{5})/2$, s is the Fibonacci word 01001010… and $q_0,q_1,q_2,\ldots$ are the Fibonacci numbers 1, 2, 3, … . Equation (6) becomes

$$q_{i+1}\leqslant m=q_{i-1}+bq_i+r<q_{i+2},\quad 1\leqslant b\leqslant a_{i+1},\ 0\leqslant r<q_i$$

If $i=0$ then $r=0$, $m=b\leqslant 2$. We are in case (iii) of Theorem 4 whence if $m=2$ $L(m)=3+2\times 2-2=5$, corresponding to factors 01010, while if $m=1$ $L(m)=2$ corresponding to factor 00.

If $i>0$ then $a_{i+1}=1$, whence $b=1=a_{i+1}$ and $m=q_{i+1}+r$, $0\leqslant r<q_i$. If $r=0$ we are in case (i) whence $L(m)=L(q_{i+1})=q_{i+2}+2q_{i+1}-2$. If $r\neq 0$ we are in no case whence $L(m)<2m$.

For instance $L(5)=8+2\times 5-2=16$, corresponding to $(01001)^{30}$, while $L(6)<12$ and indeed as can be verified $L(6)=9$, corresponding to (010010)010.

Now by Proposition 3

$$\xi(q_n)=3+q_{n-1}/q_n-2/q_n.$$

As q_{n-1}/q_n has limit $\theta=[0,\bar{1}]=(\sqrt{5}-1)/2$ we get the limit $3+(\sqrt{5}-1)/2$ for the maximal fractionary exponents. Moreover, observing that the q_{n-1}/q_n are the convergents of θ, we have $|q_{n-1}/q_n-\theta|<1/q_n^2$ whence easily $\xi(q_n)<3+\theta$. These are the results of [7].

Example 2. With $\alpha=(\sqrt{15}-3)/2=[0,\overline{2,3}]$ we have $q_1=2,\ q_2=7,\ q_3=16,\ q_4=55$, $\ldots$ and $s=0101010010101001010 1\ldots$

Here are some cases.

If $m=2$ then $i=0$, $b=2=a_1$ we are in case (iii) whence $L(m)=q_2+2q_1-2=9$, corresponding to $(01)^4 0$.

If $m=3$ then $i=1$, $b=1$, $r=0$, we are in case (i) whence $L(m)=2m+q_i-2=6$, corresponding to $(010)^2$.

If $m=4$ then $i=1$, $b=1$, $r=1$ we are in case (iv) and also find $L(4)=9$.

If $m=5$ then $i=1$, $b=2$, $r=0$, we are in case (i) whence $L(m)=2m+q_i-2=2m=10$, corresponding to $(01010)^2$.

If $m=14$ then $i=2$, $b=1$, $r=5$, we are in case (ii) whence $L(m)=q_3+2q_2-2=28=L(7)$.

If $m=6,8,10,11$ for instance we are in no case, $L(m)<2m$.

Now the exponents $\xi(m)$ to consider for getting the upper bound are the $\xi(q_{2n+1})=5+(q_{2n}-2)/q_{2n+1}$. Their limit is $5+[0,\overline{2,3}]=5+(\sqrt{15}-3)/2$ and by using explicit formulas for the q_n it is possible to see that all $\xi(m)$ are less than this value.

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